

SUPER CONGRUENCES INVOLVING BERNOULLI AND EULER POLYNOMIALS

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ABSTRACT. Let $p > 3$ be a prime, and let a be a rational p -adic integer. Let $\{B_n(x)\}$ and $\{E_n(x)\}$ denote the Bernoulli polynomials and Euler polynomials, respectively. In this paper we show that

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \equiv (-1)^{\langle a \rangle_p} + p^2 t(t+1) E_{p-3}(-a) \pmod{p^3}$$

and for $a \not\equiv -\frac{1}{2} \pmod{p}$,

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k+1} \equiv \frac{1+2t}{1+2a} + p^2 \frac{t(t+1)}{1+2a} B_{p-2}(-a) \pmod{p^3},$$

where $\langle a \rangle_p \in \{0, 1, \dots, p-1\}$ satisfying $a \equiv \langle a \rangle_p \pmod{p}$ and $t = (a - \langle a \rangle_p)/p$. Taking $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in the above congruences we solve some conjectures of Z.W. Sun. In this paper we also establish congruences for $\sum_{k=0}^{p-1} k \binom{a}{k} \binom{-1-a}{k}$, $\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k-1}$, $\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \binom{-1-a}{k} \pmod{p^3}$ and $\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{a}{k}$, $\sum_{k=0}^{p-1} \binom{a}{k} (-2)^k \pmod{p^2}$.

MSC: Primary 11A07, Secondary 11B68, 05A19

Keywords: Congruence; Euler number; Euler polynomial; Bernoulli polynomial

1. Introduction.

Let $p > 3$ be a prime. In 2003, based on his work concerning hypergeometric functions

The author is supported by the National Natural Science Foundation of China (grant no. 11371163).

and Calabi-Yau manifolds, Rodriguez-Villegas [RV] conjectured the following congruences:

$$(1.1) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$(1.2) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$(1.3) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$(1.4) \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol. These congruences were later confirmed by Mortenson [M1-M2] via the Gross-Koblitz formula. For elementary proofs of (1.1) see [S5] and [T1]. For elementary proofs of (1.2)-(1.4) see [S8].

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2) \quad \text{and} \quad B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad (n \geq 0).$$

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$E_0 = 1, \quad E_n = - \sum_{k=1}^{[n/2]} \binom{n}{2k} E_{n-2k} \quad (n \geq 1)$$

$$\text{and} \quad E_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (2x-1)^{n-k} E_k,$$

where $[a]$ is the greatest integer not exceeding a . It is well known that $B_{2n+1} = 0$ and $E_{2n-1} = 0$ for any positive integer n . $\{B_n\}$ and $\{E_n\}$ are important sequences and they have many interesting properties and applications. See [B], [MOS] and [S1,S2,S3,S4].

Let $p > 3$ be a prime and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. In [Su1], using a complicated method Z.W. Sun proved that

$$(1.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}$$

and conjectured that (see [Su1, Conjecture 5.12] and [Su2, Conjecture 1.2])

$$(1.6) \quad \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) - \frac{25}{9} p^2 E_{p-3} \pmod{p^3},$$

$$(1.7) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) - \frac{3}{16} p^2 E_{p-3} \left(\frac{1}{4}\right) \pmod{p^3},$$

$$(1.8) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) - \frac{p^2}{3} B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3},$$

$$(1.9) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k (2k+1)} \equiv \left(\frac{-1}{p}\right) - 3p^2 E_{p-3} \pmod{p^3},$$

$$(1.10) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k k} \equiv -3H_{\frac{p-1}{2}} + \frac{7}{4} p^2 B_{p-3} \pmod{p^3}.$$

As pointed out in [S8], we have

$$(1.11) \quad \begin{aligned} \left(\frac{-\frac{1}{2}}{k}\right)^2 &= \frac{\binom{2k}{k}^2}{16^k}, \quad \left(\frac{-\frac{1}{3}}{k}\right) \left(\frac{-\frac{2}{3}}{k}\right) = \frac{\binom{2k}{k} \binom{3k}{k}}{27^k}, \\ \left(\frac{-\frac{1}{4}}{k}\right) \left(\frac{-\frac{3}{4}}{k}\right) &= \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k}, \quad \left(\frac{-\frac{1}{6}}{k}\right) \left(\frac{-\frac{5}{6}}{k}\right) = \frac{\binom{3k}{k} \binom{6k}{3k}}{432^k}. \end{aligned}$$

Let \mathbb{Z} be the set of integers. For a prime p let \mathbb{Z}_p denote the set of rational p -adic integers. For a p -adic integer a let $\langle a \rangle_p \in \{0, 1, \dots, p-1\}$ be given by $a \equiv \langle a \rangle_p \pmod{p}$. Let p be an odd prime and $a \in \mathbb{Z}_p$. In [S8] the author showed that

$$(1.12) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \equiv (-1)^{\langle a \rangle_p} \pmod{p^2}.$$

For $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$, using (1.11) we get (1.1)-(1.4) immediately. In [T3] Tauraso obtained a congruence for $\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \binom{-1-a}{k} \pmod{p^2}$.

For a prime $p > 3$ and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$, in Section 2 we improve (1.12) by showing that

$$(1.13) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \equiv (-1)^{\langle a \rangle_p} + (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) E_{p-3}(-a) \pmod{p^3}.$$

Taking $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in (1.13) we deduce (1.5)-(1.8).

Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv -\frac{1}{2} \pmod{p}$. In Section 3 we prove that

$$(1.14) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k+1} \equiv \frac{1+2t}{1+2a} + p^2 \frac{t(t+1)}{1+2a} B_{p-2}(-a) \pmod{p^3},$$

where $t = (a - \langle a \rangle_p)/p$. Taking $a = -\frac{1}{4}$ in (1.14) we deduce (1.9). In Section 4 we determine $\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k-1} \pmod{p^3}$. In Section 5 we give a congruence for $\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \binom{-1-a}{k} \pmod{p^3}$. By taking $a = -\frac{1}{4}$ we get (1.10). In Section 6 we give a congruence for $\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{a}{k} \pmod{p^2}$, and in Section 7 we establish a congruence for $\sum_{k=0}^{p-1} \binom{a}{k} (-2)^k \pmod{p^2}$.

2. Congruences for $\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \pmod{p^3}$.

Lemma 2.1. *Let $p > 3$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{pt}{k} \binom{-1-pt}{k} \equiv 1 \pmod{p^3}.$$

Proof. For $k \in \{1, 2, \dots, p-1\}$ we see that

$$\begin{aligned} \binom{pt}{k} \binom{-1-pt}{k} &= \frac{pt(pt-1) \cdots (pt-k+1)(-1-pt)(-2-pt) \cdots (-k-pt)}{k!^2} \\ &= \frac{(-1)^k pt(pt+k)}{k!^2} (p^2 t^2 - 1^2) \cdots (p^2 t^2 - (k-1)^2) \\ &\equiv -\frac{pt(pt+k)}{k^2} = -\frac{p^2 t^2}{k^2} - \frac{pt}{k} \pmod{p^3}. \end{aligned}$$

From [L] or [S2] we know that

$$(2.1) \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}.$$

Thus,

$$\sum_{k=0}^{p-1} \binom{pt}{k} \binom{-1-pt}{k} \equiv 1 - p^2 t^2 \sum_{k=1}^{p-1} \frac{1}{k^2} - pt \sum_{k=1}^{p-1} \frac{1}{k} \equiv 1 \pmod{p^3}.$$

This proves the lemma.

Lemma 2.2. *Let p be an odd prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod{p}$ and $k \in \{1, 2, \dots, p-2\}$. Then*

$$\sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r^k} \equiv -\frac{(2^{p-k} - 1)B_{p-k}}{p-k} + \frac{1}{2}(-1)^{\langle a \rangle_p + k} E_{p-1-k}(-a) \pmod{p}.$$

Proof. For positive integers m and n it is well known ([MOS]) that

$$\sum_{r=0}^{m-1} (-1)^r r^n = \frac{E_n(0) - (-1)^m E_n(m)}{2}.$$

Thus,

$$\sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r^k} \equiv \sum_{r=0}^{\langle a \rangle_p} (-1)^r r^{p-1-k} = \frac{E_{p-1-k}(0) - (-1)^{\langle a \rangle_p+1} E_{p-1-k}(\langle a \rangle_p + 1)}{2} \pmod{p}.$$

From [MOS] and [S6, (2.2)-(2.3)] we know that

$$(2.2) \quad E_n(0) = \frac{2(1-2^{n+1})B_{n+1}}{n+1} \quad \text{and} \quad E_n(1-x) = (-1)^n E_n(x).$$

Hence,

$$\sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r^k} \equiv -\frac{(2^{p-k}-1)B_{p-k}}{p-k} + \frac{1}{2}(-1)^{\langle a \rangle_p+k} E_{p-1-k}(-\langle a \rangle_p) \pmod{p}.$$

Set $a = \langle a \rangle_p + pt$. It is well known ([MOS]) that $E_n(x+y) = \sum_{s=0}^n \binom{n}{s} x^s E_{n-s}(y)$. Thus,

$$\begin{aligned} E_{p-1-k}(-\langle a \rangle_p) &= E_{p-1-k}(pt-a) = \sum_{s=0}^{p-1-k} \binom{p-1-k}{s} (pt)^s E_{p-1-k-s}(-a) \\ &\equiv E_{p-1-k}(-a) \pmod{p}. \end{aligned}$$

We are done.

Theorem 2.1. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} &\equiv (-1)^{\langle a \rangle_p} + (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) E_{p-3}(-a) \\ &\equiv (-1)^{\langle a \rangle_p} + (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \left(\frac{2}{a^2} - E_{p-3}(a) \right) \pmod{p^3}. \end{aligned}$$

Proof. Set $S_{p-1}(x) = \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k}$. Then

$$S_{p-1}(a) - (-1)^{\langle a \rangle_p} S_{p-1}(a - \langle a \rangle_p) = \sum_{k=0}^{\langle a \rangle_p-1} (-1)^k (S_{p-1}(a-k) + S_{p-1}(a-k-1)).$$

Suppose $a = \langle a \rangle_p + pt$. Then $t \in \mathbb{Z}_p$ and $a-k = \langle a \rangle_p - k + pt$. For $k = 0, 1, \dots, \langle a \rangle_p - 1$ taking $m = \langle a \rangle_p - k$ and $b = -1$ in [S8, (4.3)] we see that

$$(a-k)(S_{p-1}(a-k) + S_{p-1}(a-k-1)) \equiv 2pt \cdot \frac{-p-pt}{-(\langle a \rangle_p - k)} = 2p^2 \frac{t(t+1)}{\langle a \rangle_p - k} \pmod{p^3}$$

and so

$$S_{p-1}(a-k) + S_{p-1}(a-k-1) \equiv 2p^2 t(t+1) \cdot \frac{1}{(\langle a \rangle_p - k)^2} \pmod{p^3}.$$

Therefore,

$$\begin{aligned}
S_{p-1}(a) - (-1)^{\langle a \rangle_p} S_{p-1}(pt) &= \sum_{k=0}^{\langle a \rangle_p - 1} (-1)^k (S_{p-1}(a - k) + S_{p-1}(a - k - 1)) \\
&\equiv \sum_{k=0}^{\langle a \rangle_p - 1} (-1)^k \cdot 2p^2 t(t+1) \cdot \frac{1}{(\langle a \rangle_p - k)^2} \\
&= (-1)^{\langle a \rangle_p} 2p^2 t(t+1) \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r^2} \pmod{p^3}.
\end{aligned}$$

As $B_{2m+1} = 0$ for $m \geq 1$, we see that $B_{p-2} = 0$. Thus, by Lemma 2.2 we have

$$\sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r^2} \equiv \frac{1}{2} (-1)^{\langle a \rangle_p} E_{p-3}(-a) \pmod{p}.$$

Now, from the above and Lemma 2.1 we deduce that

$$\begin{aligned}
S_{p-1}(a) &\equiv (-1)^{\langle a \rangle_p} S_{p-1}(pt) + (-1)^{\langle a \rangle_p} 2p^2 t(t+1) \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r^2} \\
&\equiv (-1)^{\langle a \rangle_p} + p^2 t(t+1) E_{p-3}(-a) \pmod{p^3}.
\end{aligned}$$

It is well known that ([MOS]) $E_n(1-x) = (-1)^n E_n(x)$ and $E_n(x) + E_n(x+1) = 2x^n$. Thus,

$$E_{p-3}(-a) = E_{p-3}(1+a) = 2a^{p-3} - E_{p-3}(a) \equiv \frac{2}{a^2} - E_{p-3}(a) \pmod{p}.$$

Recall that $t = (a - \langle a \rangle_p)/p$. By the above, the theorem is proved.

Taking $a = -\frac{1}{2}$ in Theorem 2.1 and then applying (1.11) and the fact $E_n = 2^n E_n(\frac{1}{2})$ we obtain (1.5).

For $m = 3, 4, 6$ it is clear that

$$(2.3) \quad -\frac{1}{m} - \langle -\frac{1}{m} \rangle_p = \begin{cases} -\frac{1}{m} - \frac{p-1}{m} = -\frac{p}{m} & \text{if } p \equiv 1 \pmod{m}, \\ -\frac{1}{m} - \frac{(m-1)p-1}{m} = -\frac{(m-1)p}{m} & \text{if } p \equiv -1 \pmod{m} \end{cases}$$

and so

$$(2.4) \quad \left(-\frac{1}{m} - \langle -\frac{1}{m} \rangle_p \right) \left(p - \frac{1}{m} - \langle -\frac{1}{m} \rangle_p \right) = -\frac{p}{m} \cdot \frac{(m-1)p}{m} = -\frac{m-1}{m^2} p^2.$$

Corollary 2.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) - \frac{25}{9} p^2 E_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{6}$ in Theorem 2.1 and then applying (1.11) and (2.4) we see that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} &= \sum_{k=0}^{p-1} \binom{-\frac{1}{6}}{k} \binom{-\frac{5}{6}}{k} \\ &\equiv (-1)^{\langle -\frac{1}{6} \rangle_p} + \left(-\frac{1}{6} - \langle -\frac{1}{6} \rangle_p\right) \left(p - \frac{1}{6} - \langle -\frac{1}{6} \rangle_p\right) E_{p-3} \left(\frac{1}{6}\right) \\ &\equiv \left(\frac{-1}{p}\right) - \frac{5}{36} E_{p-3} \left(\frac{1}{6}\right) \pmod{p^3}. \end{aligned}$$

By [S6, Theorem 2.1 and Lemma 2.1], we have $6^{2n} E_{2n}(\frac{1}{6}) = \frac{3^{2n}+1}{2} E_{2n}$. Thus, $E_{p-3}(\frac{1}{6}) = \frac{1}{6^{p-3}} \cdot \frac{3^{p-3}+1}{2} E_{p-3} \equiv 20 E_{p-3} \pmod{p}$. Hence the result follows.

In [S7] the author introduced the sequence $\{U_n\}$ given by

$$U_0 = 1, \quad U_n = -2 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k} \quad (n \geq 1)$$

or

$$\sum_{n=0}^{\infty} U_n \frac{t^n}{n!} = \frac{1}{e^t + e^{-t} - 1} \quad (|t| < \frac{\pi}{3}).$$

Clearly $U_{2n-1} = 0$. The first few values of U_{2n} are shown below:

$$\begin{aligned} U_2 &= -2, & U_4 &= 22, & U_6 &= -602, & U_8 &= 30742, & U_{10} &= -2523002, \\ U_{12} &= 303692662, & U_{14} &= -50402079002, & U_{16} &= 11030684333782. \end{aligned}$$

For any prime $p > 3$, in [S7] the author proved that

$$\sum_{k=1}^{\lfloor 2p/3 \rfloor} \frac{(-1)^{k-1}}{k} \equiv 3p \left(\frac{p}{3}\right) U_{p-3} \pmod{p^2}.$$

Corollary 2.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) - 2p^2 U_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 2.1 and then applying (1.11) and (2.4) we see that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} &= \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{2}{3}}{k} \\ &\equiv (-1)^{\langle -\frac{1}{3} \rangle_p} + \left(-\frac{1}{3} - \langle -\frac{1}{3} \rangle_p \right) \left(p - \frac{1}{3} - \langle -\frac{1}{3} \rangle_p \right) E_{p-3} \left(\frac{1}{3} \right) \\ &= \left(\frac{-3}{p} \right) - \frac{2}{9} E_{p-3} \left(\frac{1}{3} \right) \pmod{p^3}. \end{aligned}$$

By [S7, Theorem 2.1], $U_{2n} = 3^{2n} E_{2n}(\frac{1}{3})$. Thus, $U_{p-3} = 3^{p-3} E_{p-3}(\frac{1}{3}) \equiv \frac{1}{9} E_{p-3}(\frac{1}{3}) \pmod{p}$. Now putting all the above together we obtain the result.

Remark 2.1 Let $p > 3$ be a prime. By [S7, p.217], $B_{p-2}(\frac{1}{3}) \equiv 6U_{p-3} \pmod{p}$. Thus, from Corollary 2.2 we deduce (1.8). In [MT], Mattarei and Tauraso proved that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{-3}{p} \right) - \frac{p^2}{3} B_{p-2} \left(\frac{1}{3} \right) \pmod{p^3}.$$

This together with Corollary 2.2 yields

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{-3}{p} \right) - 2p^2 U_{p-3} \pmod{p^3}.$$

In [S3] the author introduced the sequence $\{S_n\}$ given by

$$S_0 = 1 \quad \text{and} \quad S_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} 2^{2n-2k-1} S_k \quad (n \geq 1),$$

and showed that $S_n = 4^n E_n(\frac{1}{4})$.

Corollary 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p} \right) - 3p^2 S_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{4}$ in Theorem 2.1 and then applying (1.11) and (2.4) we see that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k} &= \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{3}{4}}{k} \\ &\equiv (-1)^{\langle -\frac{1}{4} \rangle_p} + \left(-\frac{1}{4} - \langle -\frac{1}{4} \rangle_p \right) \left(p - \frac{1}{4} - \langle -\frac{1}{4} \rangle_p \right) E_{p-3} \left(\frac{1}{4} \right) \\ &= \left(\frac{-2}{p} \right) - \frac{3}{16} E_{p-3} \left(\frac{1}{4} \right) \pmod{p^3}. \end{aligned}$$

Since $S_{p-3} = 4^{p-3} E_{p-3}(\frac{1}{4}) \equiv \frac{1}{16} E_{p-3}(\frac{1}{4}) \pmod{p}$, we obtain the result.

Corollary 2.4. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} + \sum_{k=0}^{p-1} \binom{-a}{k} \binom{-1+a}{k} \equiv (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \frac{2}{a^2} \pmod{p^3}.$$

Proof. As $\langle -a \rangle_p = p - \langle a \rangle_p$, from Theorem 2.1 we derive that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-a}{k} \binom{-1+a}{k} &\equiv (-1)^{\langle -a \rangle_p} + (-a - \langle -a \rangle_p)(p - a - \langle -a \rangle_p) \left(\frac{2}{a^2} - E_{p-3}(-a) \right) \\ &= -(-1)^{\langle a \rangle_p} + (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \left(\frac{2}{a^2} - E_{p-3}(-a) \right) \\ &\equiv (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \frac{2}{a^2} - \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \pmod{p^3}. \end{aligned}$$

This yields the result.

Lemma 2.3. *For any nonnegative integer n we have*

$$\sum_{k=0}^n (k - a(a+1)) \binom{a}{k} \binom{-1-a}{k} = -a(a+1) \binom{a-1}{n} \binom{-2-a}{n}.$$

Proof. Observe that

$$\begin{aligned} &-a(a+1) \left\{ \binom{a-1}{n+1} \binom{-2-a}{n+1} - \binom{a-1}{n} \binom{-2-a}{n} \right\} \\ &= \binom{a}{n+1} \binom{-1-a}{n+1} ((a-n-1)(-2-a-n) - (n+1)^2) \\ &= (n+1 - a(a+1)) \binom{a}{n+1} \binom{-1-a}{n+1}. \end{aligned}$$

The result can be easily proved by induction on n .

Theorem 2.2. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0, -1 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} k \binom{a}{k} \binom{-1-a}{k} \equiv (-1)^{\langle a \rangle_p} a(a+1) + p^2 t(t+1) (a(a+1) E_{p-3}(-a) - 1) \pmod{p^3},$$

where $t = (a - \langle a \rangle_p)/p$.

Proof. By [S8, Lemma 4.2],

$$\begin{aligned} \binom{a-1}{p-1} &= \binom{\langle a \rangle_p + pt - 1}{p-1} \equiv \frac{pt}{\langle a \rangle_p} \pmod{p^2}, \\ \binom{-2-a}{p-1} &= \binom{p-1 - \langle a \rangle_p - p(t+1) - 1}{p-1} \equiv \frac{p(-t-1)}{p-1 - \langle a \rangle_p} \equiv \frac{p(t+1)}{\langle a \rangle_p + 1} \pmod{p^2}. \end{aligned}$$

Thus,

$$\binom{a-1}{p-1} \binom{-2-a}{p-1} \equiv \frac{t(t+1)}{\langle a \rangle_p (\langle a \rangle_p + 1)} p^2 \equiv \frac{t(t+1)}{a(a+1)} p^2 \pmod{p^3}.$$

Hence, using Lemma 2.3 we see that

$$\begin{aligned} & \sum_{k=0}^{p-1} k \binom{a}{k} \binom{-1-a}{k} - a(a+1) \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \\ &= -a(a+1) \binom{a-1}{p-1} \binom{-2-a}{p-1} \equiv -p^2 t(t+1) \pmod{p^3}. \end{aligned}$$

This together with Theorem 2.1 yields the result.

Corollary 2.5. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{k \binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv -\frac{5}{36} \left(\frac{-1}{p} \right) + p^2 \left(\frac{5}{36} + \frac{125}{324} E_{p-3} \right) \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{6}$ in Theorem 2.2 and then applying (1.11) we see that

$$\sum_{k=0}^{p-1} \frac{k \binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv -\frac{5}{36} \left(\frac{-1}{p} \right) - \frac{5}{36} p^2 \left(-\frac{5}{36} E_{p-3} \left(\frac{1}{6} \right) - 1 \right) \pmod{p^3}.$$

By the proof of Corollary 2.1, $E_{p-3}(\frac{1}{6}) \equiv 20E_{p-3} \pmod{p}$. Thus the result follows.

Corollary 2.6. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{k \binom{2k}{k} \binom{3k}{k}}{27^k} \equiv -\frac{2}{9} \left(\frac{-3}{p} \right) + p^2 \left(\frac{2}{9} + \frac{4}{9} U_{p-3} \right) \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 2.2 and then applying (1.11) and the fact $E_{p-3}(\frac{1}{3}) \equiv 9U_{p-3} \pmod{p}$ we deduce the result.

3. Congruences for $\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k+1} \pmod{p^3}$.

For any positive integer n and variables a and b with $b \notin \{-1, -\frac{1}{2}, \dots, -\frac{1}{n}\}$ let

$$(3.1) \quad S_n(a, b) = \sum_{k=0}^n \binom{a}{k} \binom{-1-a}{k} \frac{1}{bk+1}.$$

Then

$$\begin{aligned} & (ab+1)S_n(a, b) - (ab-1)S_n(a-1, b) \\ &= \sum_{k=0}^n \binom{a}{k} \binom{-1-a}{k} \frac{ab+1}{bk+1} - \sum_{k=0}^n \binom{a-1}{k} \binom{-a}{k} \frac{ab-1}{bk+1} \\ &= \sum_{k=0}^n \binom{a}{k} \binom{-a}{k} \left(\frac{ab+1}{bk+1} \cdot \frac{a+k}{a} - \frac{ab-1}{bk+1} \cdot \frac{a-k}{a} \right) \\ &= 2 \sum_{k=0}^n \binom{a}{k} \binom{-a}{k}. \end{aligned}$$

By [S8, (4.5)] or induction on n ,

$$\sum_{k=0}^n \binom{a}{k} \binom{-a}{k} = \binom{n+a}{n} \binom{n-a}{n} = \binom{a-1}{n} \binom{-a-1}{n}.$$

Thus,

$$(3.2) \quad (ab+1)S_n(a, b) - (ab-1)S_n(a-1, b) = 2 \binom{a-1}{n} \binom{-a-1}{n}.$$

Lemma 3.1 ([S8, Lemma 4.2]). *Let p be an odd prime, $m \in \{1, 2, \dots, p-1\}$ and $t \in \mathbb{Z}_p$. Then*

$$\binom{m+pt-1}{p-1} \equiv \frac{pt}{m} - \frac{p^2 t^2}{m^2} + \frac{p^2 t}{m} H_m \pmod{p^3}.$$

Lemma 3.2. *Let p be an odd prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$. Then*

$$\binom{a-1}{p-1} \binom{-a-1}{p-1} \equiv p^2 \frac{t(t+1)}{\langle a \rangle_p^2} + p^3 t(t+1) \left(-\frac{1+2t}{a^3} + 2 \frac{H_{\langle a \rangle_p}}{a^2} \right) \pmod{p^4},$$

where $t = (a - \langle a \rangle_p)/p$.

Proof. By Lemma 3.1,

$$\begin{aligned} \binom{a-1}{p-1} &= \binom{\langle a \rangle_p + pt - 1}{p-1} \equiv \frac{pt}{\langle a \rangle_p} + p^2 t \left(-\frac{t}{\langle a \rangle_p^2} + \frac{1}{\langle a \rangle_p} H_{\langle a \rangle_p} \right) \\ &\equiv \frac{pt}{\langle a \rangle_p} + p^2 t \left(-\frac{t}{a^2} + \frac{H_{\langle a \rangle_p}}{a} \right) \pmod{p^3}. \end{aligned}$$

From [S8, p.312] we know that $H_{p-1-\langle a \rangle_p} \equiv H_{\langle a \rangle_p} \pmod{p}$. Thus, from Lemma 3.1 we deduce that

$$\begin{aligned} \binom{-a-1}{p-1} &= \binom{p - \langle a \rangle_p - p(t+1) - 1}{p-1} \\ &\equiv \frac{p(-t-1)}{p - \langle a \rangle_p} + p^2(-t-1) \left(-\frac{-t-1}{(p - \langle a \rangle_p)^2} + \frac{H_{p-\langle a \rangle_p}}{p - \langle a \rangle_p} \right) \\ &\equiv \frac{p(t+1)(\langle a \rangle_p + p)}{\langle a \rangle_p^2} - p^2(t+1) \left(\frac{t+1}{\langle a \rangle_p^2} - \frac{H_{p-\langle a \rangle_p}}{\langle a \rangle_p} \right) \\ &\equiv \frac{p(t+1)}{\langle a \rangle_p} + p^2(t+1) \left\{ \frac{1}{\langle a \rangle_p^2} - \frac{t+1}{\langle a \rangle_p^2} + \frac{-\frac{1}{\langle a \rangle_p} + H_{p-1-\langle a \rangle_p}}{\langle a \rangle_p} \right\} \\ &\equiv \frac{p(t+1)}{\langle a \rangle_p} + p^2(t+1) \left\{ -\frac{1+t}{\langle a \rangle_p^2} + \frac{H_{\langle a \rangle_p}}{\langle a \rangle_p} \right\} \\ &\equiv \frac{p(t+1)}{\langle a \rangle_p} + p^2(t+1) \left(-\frac{1+t}{a^2} + \frac{H_{\langle a \rangle_p}}{a} \right) \pmod{p^3}. \end{aligned}$$

Hence,

$$\begin{aligned}
& \binom{a-1}{p-1} \binom{-a-1}{p-1} \\
& \equiv \left(\frac{pt}{\langle a \rangle_p} + p^2 t \left(-\frac{t}{a^2} + \frac{H_{\langle a \rangle_p}}{a} \right) \right) \left(\frac{p(t+1)}{\langle a \rangle_p} + p^2(t+1) \left(-\frac{1+t}{a^2} + \frac{H_{\langle a \rangle_p}}{a} \right) \right) \\
& \equiv p^2 \frac{t(t+1)}{\langle a \rangle_p^2} + p^3 t(t+1) \left(-\frac{1+2t}{a^3} + 2 \frac{H_{\langle a \rangle_p}}{a^2} \right) \pmod{p^4}.
\end{aligned}$$

This proves the lemma.

For any positive integer n and variable a let

$$(3.3) \quad T_n(a) = (2a+1)S_n(a, 2) = \sum_{k=0}^n \binom{a}{k} \binom{-1-a}{k} \frac{2a+1}{2k+1}.$$

Lemma 3.3. *Let $p > 3$ be a prime and $t \in \mathbb{Z}_p$. Then $T_{p-1}(pt) \equiv 1 + 2t \pmod{p^3}$.*

Proof. Clearly

$$\begin{aligned}
T_{p-1}(pt) &= \sum_{k=0}^{p-1} \binom{pt}{k} (-1)^k \binom{pt+k}{k} \frac{2pt+1}{2k+1} \\
&= 2pt+1 + \sum_{k=1}^{p-1} \frac{(-1)^k pt(pt+k)(p^2t^2 - (k-1)^2) \cdots (p^2t^2 - 1^2)}{k!^2} \cdot \frac{2pt+1}{2k+1} \\
&\equiv 2pt+1 + \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} pt(pt+k) \frac{(-1)^k (-1^2)(-2^2) \cdots (-(k-1)^2)}{k!^2} \cdot \frac{2pt+1}{2k+1} \\
&\quad + (-1)^{\frac{p-1}{2}} \frac{(2pt+1)t}{pt - \frac{p-1}{2}} \cdot \frac{(p^2t^2 - (\frac{p-1}{2})^2) \cdots (p^2t^2 - 1^2)}{(\frac{p-1}{2}!)^2} \\
&\equiv 2pt+1 - pt(2pt+1) \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{pt+k}{k^2(2k+1)} + \frac{2t(2pt+1)}{2pt+1-p} \left(1 - p^2t^2 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2} \right) \pmod{p^3}.
\end{aligned}$$

As $\frac{1}{k^2(2k+1)} = \frac{1}{k^2} - \frac{2}{k} + \frac{4}{2k+1}$, using (2.1) we see that

$$\begin{aligned}
\sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{pt+k}{k^2(2k+1)} &= \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} (pt+k) \left(\frac{1}{k^2} - \frac{2}{k} + \frac{4}{2k+1} \right) \\
&= \sum_{k=1}^{p-1} (pt+k) \left(\frac{1}{k^2} - \frac{2}{k} \right) - \left(pt + \frac{p-1}{2} \right) \left(\frac{1}{(\frac{p-1}{2})^2} - \frac{2}{\frac{p-1}{2}} \right) + 2 \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{2pt-1+2k+1}{2k+1} \\
&\equiv pt \sum_{k=1}^{p-1} \left(\frac{1}{k^2} - 2\frac{1}{k} \right) + \sum_{k=1}^{p-1} \left(\frac{1}{k} - 2 \right) - pt \left(\frac{1}{\frac{1}{4}} - \frac{2}{-\frac{1}{2}} \right) \\
&\quad - \left(\frac{1}{\frac{p-1}{2}} - 2 \right) + 2 \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} 1 + 2(2pt-1) \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{1}{2k+1} \\
&\equiv -8pt + 2(p+1) + 2(2pt-1) \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{1}{2k+1} \pmod{p^2}.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{1}{2k+1} &= \sum_{k=1}^{\frac{p-3}{2}} \left(\frac{1}{2k+1} + \frac{1}{2(p-k)+1} \right) + \frac{1}{2 \cdot \frac{p+1}{2} + 1} \\
&= \sum_{k=1}^{\frac{p-3}{2}} \left(\frac{1}{2k+1} + \frac{2p+2k-1}{(2p)^2 - (2k-1)^2} \right) + \frac{1}{p+2} \\
&\equiv -2p \sum_{k=1}^{\frac{p-3}{2}} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\frac{p-3}{2}} \left(\frac{1}{2k+1} - \frac{1}{2k-1} \right) + \frac{1}{p+2} \\
&= -2p \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{(2k-1)^2} + 2p \cdot \frac{1}{(p-2)^2} - 1 + \frac{1}{p-2} + \frac{1}{p+2} \\
&\equiv -2p \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{(2k-1)^2} - 1 \pmod{p^2}.
\end{aligned}$$

By [S3, Corollary 2.1],

$$\begin{aligned}
\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{(2k-1)^2} &= \sum_{\substack{x=1 \\ x \equiv 1 \pmod{2}}}^{p-1} \frac{1}{x^2} \equiv \sum_{\substack{x=0 \\ x \equiv 1 \pmod{2}}}^{p-1} x^{p-3} \\
&\equiv \frac{2^{p-3}}{p-2} (B_{p-2}(0) - B_{p-2}(0)) = 0 \pmod{p}.
\end{aligned}$$

Hence,

$$(3.4) \quad \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{1}{2k+1} \equiv -1 \pmod{p^2}.$$

Therefore,

$$\sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{pt+k}{k^2(2k+1)} \equiv -8pt + 2(p+1) - 2(2pt-1) = 2p(1-6t) + 4 \pmod{p^2}.$$

By [S2, Corollary 5.2], $\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv 0 \pmod{p}$. Thus, from all the above we deduce that

$$\begin{aligned} T_{p-1}(pt) &\equiv 2pt + 1 - pt(2pt+1) \sum_{\substack{k=1 \\ k \neq \frac{p-1}{2}}}^{p-1} \frac{pt+k}{k^2(2k+1)} + \frac{2t(2pt+1)}{2pt+1-p} \\ &\equiv 2pt + 1 - pt(2pt+1)(2p(1-6t) + 4) + 2t \left(1 + \frac{p}{1 + (2t-1)p} \right) \\ &\equiv 2pt + 1 - pt(8pt + 2p(1-6t) + 4) + 2t + 2tp(1 - (2t-1)p) \\ &= 1 + 2t \pmod{p^3}. \end{aligned}$$

This proves the lemma.

Theorem 3.1. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a(2a+1) \not\equiv 0 \pmod{p}$. Then*

$$\begin{aligned} &\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k+1} \\ &\equiv \frac{1+2t}{1+2a} + p^2 \frac{t(t+1)}{1+2a} B_{p-2}(-a) \equiv \frac{1+2t}{1+2a} + p^2 \frac{t(t+1)}{1+2a} \left(\frac{2}{a^2} - B_{p-2}(a) \right) \pmod{p^3}, \end{aligned}$$

where $t = (a - \langle a \rangle_p)/p$.

Proof. As $a \not\equiv \frac{p-1}{2} \pmod{p}$, we see that

$$\begin{aligned} \binom{a}{\frac{p-1}{2}} \binom{-1-a}{\frac{p-1}{2}} &= (-1)^{\frac{p-1}{2}} \binom{a}{\frac{p-1}{2}} \binom{a + \frac{p-1}{2}}{\frac{p-1}{2}} \\ &= (-1)^{\frac{p-1}{2}} \frac{(a + \frac{p-1}{2})(a + \frac{p-1}{2} - 1) \cdots (a - \frac{p-1}{2} + 1)}{(\frac{p-1}{2}!)^2} \equiv 0 \pmod{p}. \end{aligned}$$

Thus, $\binom{a}{k} \binom{-1-a}{2k+1} \frac{1}{2k+1} \in \mathbb{Z}_p$ for $k = 0, 1, \dots, p-1$. Let $T_n(a)$ be given by (3.3). By (3.2) and Lemma 3.2 we have

$$(3.5) \quad \begin{aligned} &T_{p-1}(a) - T_{p-1}(a-1) \\ &= 2 \binom{a-1}{p-1} \binom{-a-1}{p-1} \equiv 2p^2 \frac{t(t+1)}{\langle a \rangle_p^2} + 2p^3 t(t+1) \left(-\frac{1+2t}{a^3} + 2 \frac{H_{\langle a \rangle_p}}{a^2} \right) \pmod{p^4}. \end{aligned}$$

For $1 \leq k \leq \langle a \rangle_p$ we have $\langle a - k + 1 \rangle_p = \langle a \rangle_p - k + 1$ and so $a - k + 1 = \langle a \rangle_p - k + 1 + pt = \langle a - k + 1 \rangle_p + pt$. Hence

$$\begin{aligned} T_{p-1}(a) - T_{p-1}(a - \langle a \rangle_p) &= \sum_{k=1}^{\langle a \rangle_p} (T_{p-1}(a - k + 1) - T_{p-1}(a - k)) \\ &\equiv \sum_{k=1}^{\langle a \rangle_p} \frac{2t(t+1)p^2}{\langle a - k + 1 \rangle_p^2} = 2t(t+1)p^2 \sum_{k=1}^{\langle a \rangle_p} \frac{1}{(\langle a \rangle_p - k + 1)^2} \\ &= 2t(t+1)p^2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} \equiv 2t(t+1)p^2 \sum_{r=1}^{\langle a \rangle_p} r^{p-3} \pmod{p^3}. \end{aligned}$$

By [S2, Lemma 3.2],

$$(3.6) \quad \sum_{r=1}^{\langle a \rangle_p} r^{p-3} \equiv (-1)^{p-2} \frac{B_{p-2}(-a) - B_{p-2}}{p-2} \equiv \frac{1}{2} B_{p-2}(-a) \pmod{p}.$$

Thus,

$$\begin{aligned} T_{p-1}(a) - T_{p-1}(pt) &= T_{p-1}(a) - T_{p-1}(a - \langle a \rangle_p) \\ &\equiv 2t(t+1)p^2 \cdot \frac{1}{2} B_{p-2}(-a) = p^2 t(t+1) B_{p-2}(-a) \pmod{p^3}. \end{aligned}$$

This together with Lemma 3.3 yields $T_{p-1}(a) \equiv 1 + 2t + p^2 t(t+1) B_{p-2}(-a) \pmod{p^3}$. From [MOS] we know that $B_n(-a) = (-1)^n (B_n(a) + na^{n-1})$. Thus,

$$B_{p-2}(-a) = (-1)^{p-2} (B_{p-2}(a) + (p-2)a^{p-3}) \equiv -B_{p-2}(a) + \frac{2}{a^2} \pmod{p}.$$

This completes the proof.

Corollary 3.1. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0, \pm \frac{1}{2} \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{2a+1}{2k+1} + \sum_{k=0}^{p-1} \binom{-a}{k} \binom{-1+a}{k} \frac{1-2a}{2k+1} \equiv (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \frac{2}{a^2} \pmod{p^3}.$$

Proof. As $\langle -a \rangle_p = p - \langle a \rangle_p$, from Theorem 3.1 we derive that

$$\begin{aligned} &\sum_{k=0}^{p-1} \binom{-a}{k} \binom{-1+a}{k} \frac{1-2a}{2k+1} \\ &\equiv 1 + 2 \frac{-a - \langle -a \rangle_p}{p} + (-a - \langle -a \rangle_p)(p - a - \langle -a \rangle_p) \left(\frac{2}{a^2} - B_{p-2}(-a) \right) \\ &= -1 - 2 \frac{a - \langle a \rangle_p}{p} + (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \left(\frac{2}{a^2} - B_{p-2}(-a) \right) \\ &\equiv (a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \frac{2}{a^2} - \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{2a+1}{2k+1} \pmod{p^3}. \end{aligned}$$

This yields the result.

Theorem 3.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k (2k+1)} \equiv (-1)^{\frac{p-1}{2}} - 3p^2 E_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{4}$ in Theorem 3.1 and then applying (1.11), (2.3) and (2.4) we obtain

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k (2k+1)} &\equiv \frac{1 + 2(-\frac{2 - (\frac{-1}{p})}{4})}{1 + 2(-\frac{1}{4})} + p^2 \frac{-\frac{1}{4} \cdot \frac{3}{4}}{1 + 2(-\frac{1}{4})} B_{p-2}\left(\frac{1}{4}\right) \\ &= (-1)^{\frac{p-1}{2}} - \frac{3}{8} p^2 B_{p-2}\left(\frac{1}{4}\right) \pmod{p^3}. \end{aligned}$$

It is known (see for example [S4, Lemma 2.5]) that $E_{2n} = -4^{2n+1} \frac{B_{2n+1}(\frac{1}{4})}{2n+1}$. Thus,

$$E_{p-3} = -4^{p-2} \frac{B_{p-2}(\frac{1}{4})}{p-2} \equiv \frac{B_{p-2}(\frac{1}{4})}{8} \pmod{p}.$$

Now combining all the above we obtain the result.

Theorem 3.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k (2k+1)} \equiv \left(\frac{p}{3}\right) - 4p^2 U_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 3.1 and then applying (1.11), (2.3) and (2.4) we obtain

$$(3.7) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k (2k+1)} \equiv \left(\frac{p}{3}\right) - \frac{2}{3} p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

By [S7, p.217], $B_{p-2}(\frac{1}{3}) \equiv 6U_{p-3} \pmod{p}$. Thus the result follows.

Corollary 3.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} (4k+1)}{27^k (2k+1)} \equiv \left(\frac{p}{3}\right) \pmod{p^3}.$$

Proof. As $2 - \frac{1}{2k+1} = \frac{4k+1}{2k+1}$, combining Corollary 2.2 with Theorem 3.3 we deduce the result.

Theorem 3.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k (2k+1)} \equiv \left(\frac{p}{3}\right) - \frac{25}{4} p^2 U_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{6}$ in Theorem 3.1 and then applying (1.11), (2.3) and (2.4) we obtain

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k (2k+1)} \equiv \left(\frac{p}{3}\right) - \frac{5}{24} p^2 B_{p-2} \left(\frac{1}{6}\right) \pmod{p^3}.$$

By [S7, p.216], $B_{p-2}(\frac{1}{6}) \equiv 30U_{p-3} \pmod{p}$. Thus the result follows.

Remark 3.1 Corollary 3.2, (3.7) and the congruence $\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k (2k+1)} \equiv \left(\frac{p}{3}\right) \pmod{p^2}$ were conjectured by Z.W. Sun in [Su2].

Theorem 3.5. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $2a+1 \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{a+k+1}{2k+1} \equiv \frac{1+(-1)^{\langle a \rangle_p}}{2} + t + p^2 \frac{t(t+1)}{4} B_{p-2} \left(-\frac{a}{2}\right) \pmod{p^3},$$

where $t = (a - \langle a \rangle_p)/p$.

Proof. From [MOS] we know that $E_n(x) = \frac{2}{n+1} (B_{n+1}(x) - 2^{n+1} B_{n+1}(\frac{x}{2}))$. Thus,

$$E_{p-3}(-a) = \frac{2}{p-2} \left(B_{p-2}(-a) - 2^{p-2} B_{p-2} \left(-\frac{a}{2}\right) \right) \equiv -B_{p-2}(-a) + \frac{1}{2} B_{p-2} \left(-\frac{a}{2}\right) \pmod{p}.$$

Now from the above and Theorems 2.1 and 3.1 we deduce that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \left(1 + \frac{2a+1}{2k+1}\right) \\ & \equiv (-1)^{\langle a \rangle_p} + p^2 t(t+1) E_{p-3}(-a) + 1 + 2t + p^2 t(t+1) B_{p-2}(-a) \\ & \equiv 1 + (-1)^{\langle a \rangle_p} + 2t + p^2 \frac{t(t+1)}{2} B_{p-2} \left(-\frac{a}{2}\right) \pmod{p^3}. \end{aligned}$$

This yields the result.

4. Congruences for $\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k-1} \pmod{p^3}$.

Lemma 4.1. *For any nonnegative integer n we have*

$$\sum_{k=0}^n \binom{a}{k} \binom{-1-a}{k} \frac{(2a(a+1)+1)k - a(a+1)}{4k^2 - 1} = \frac{a(a+1)}{2n+1} \binom{a-1}{n} \binom{-2-a}{n}.$$

Proof. It is easy to check that

$$\begin{aligned}
& \frac{a(a+1)}{2(n+1)+1} \binom{a-1}{n+1} \binom{-2-a}{n+1} - \frac{a(a+1)}{2n+1} \binom{a-1}{n} \binom{-2-a}{n} \\
&= \binom{a}{n+1} \binom{-1-a}{n+1} \left\{ \frac{a(a+1)}{2n+3} \cdot \frac{a-n-1}{a} \cdot \frac{-2-a-n}{-1-a} - \frac{a(a+1)}{2n+1} \cdot \frac{n+1}{a} \cdot \frac{n+1}{-1-a} \right\} \\
&= \binom{a}{n+1} \binom{-1-a}{n+1} \frac{(2a(a+1)+1)(n+1) - a(a+1)}{4(n+1)^2 - 1}.
\end{aligned}$$

Thus the result can be easily proved by induction on n .

Lemma 4.2. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0, -1, \pm \frac{1}{2} \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{(2a(a+1)+1)k - a(a+1)}{4k^2 - 1} \equiv -(a - \langle a \rangle_p)(p + a - \langle a \rangle_p) \pmod{p^3}.$$

Proof. Set $a = \langle a \rangle_p + pt$. By Lemma 3.1,

$$\begin{aligned}
\binom{a-1}{p-1} &= \binom{\langle a \rangle_p + pt - 1}{p-1} \equiv \frac{pt}{\langle a \rangle_p} \equiv \frac{pt}{a} \pmod{p^2}, \\
\binom{-2-a}{p-1} &= \binom{p-1 - \langle a \rangle_p - p(t+1) - 1}{p-1} \equiv \frac{-p(t+1)}{p-1 - \langle a \rangle_p} \equiv \frac{p(t+1)}{a+1} \pmod{p^2}.
\end{aligned}$$

Thus,

$$\frac{a(a+1)}{2(p-1)+1} \binom{a-1}{p-1} \binom{-2-a}{p-1} \equiv \frac{a(a+1)}{2p-1} \cdot \frac{pt}{a} \cdot \frac{p(t+1)}{a+1} \equiv -p^2 t(t+1) \pmod{p^3}.$$

Now taking $n = p-1$ in Lemma 4.1 and then applying the above we obtain the result.

Theorem 4.1. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0, -1, \pm \frac{1}{2} \pmod{p}$. Then*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2k-1} \\
& \equiv -(2a+1)(2t+1) - p^2 t(t+1)(4 + (2a+1)B_{p-2}(-a)) \pmod{p^3},
\end{aligned}$$

where $t = (a - \langle a \rangle_p)/p$.

Proof. Note that $\frac{1}{2k-1} = 4 \frac{(2a(a+1)+1)k - a(a+1)}{4k^2 - 1} - \frac{(2a+1)^2}{2k+1}$. Combining Theorem 3.1 with Lemma 4.2 we deduce the result.

Corollary 4.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k (2k-1)} \equiv -\frac{1}{4} \left(\frac{-1}{p} \right) + \frac{3}{4} p^2 (1 + E_{p-3}) \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{4}$ in Theorem 4.1 and noting that $B_{p-2}(\frac{1}{4}) \equiv 8E_{p-3} \pmod{p}$ we deduce the result.

Corollary 4.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k (2k-1)} \equiv -\frac{1}{9} \left(\frac{-3}{p} \right) + \frac{4}{9} p^2 (2 - U_{p-3}) \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 4.1 and noting that $B_{p-2}(\frac{1}{3}) \equiv 6U_{p-3} \pmod{p}$ we deduce the result.

Corollary 4.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k (2k-1)} \equiv -\frac{4}{9} \left(\frac{-3}{p} \right) + \frac{5}{9} p^2 (1 + 5U_{p-3}) \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{6}$ in Theorem 4.1 and noting that $B_{p-2}(\frac{1}{6}) \equiv 30U_{p-3} \pmod{p}$ we deduce the result.

5. Congruences for $\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \binom{-1-a}{k} \pmod{p^3}$.

For any positive integer n and variable a let

$$A_n(a) = \sum_{k=1}^n \binom{a}{k} \binom{-1-a}{k} \frac{1}{k}.$$

Then

$$\begin{aligned} A_n(a) - A_n(a-1) &= \sum_{k=1}^n \frac{1}{k} \left\{ \binom{a}{k} \binom{-1-a}{k} - \binom{a-1}{k} \binom{-a}{k} \right\} \\ &= \sum_{k=1}^n \frac{1}{k} \binom{a}{k} \binom{-a}{k} \left(\frac{a+k}{a} - \frac{a-k}{a} \right) \\ &= \frac{2}{a} \sum_{k=1}^n \binom{a}{k} \binom{-a}{k}. \end{aligned}$$

By [S7, (4.5)] or induction on n ,

$$\sum_{k=0}^n \binom{a}{k} \binom{-a}{k} = \binom{n+a}{n} \binom{n-a}{n} = \binom{a-1}{n} \binom{-a-1}{n}.$$

Thus,

$$(5.1) \quad A_n(a) - A_n(a-1) = \frac{2}{a} \binom{a-1}{n} \binom{-a-1}{n} - \frac{2}{a}.$$

Hence, if $p > 3$ is a prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$, by (5.1) and Lemma 3.2 we have

$$(5.2) \quad A_{p-1}(a) - A_{p-1}(a-1) \equiv \frac{2t(t+1)}{a\langle a \rangle_p^2} p^2 - \frac{2}{a} \equiv \frac{2t(t+1)}{\langle a \rangle_p^3} p^2 - \frac{2}{a} \pmod{p^3},$$

where $t = (a - \langle a \rangle_p)/p$.

Lemma 5.1. *Let $p > 3$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$\sum_{k=1}^{p-1} \binom{pt}{k} \binom{-1-pt}{k} \frac{1}{k} \equiv -\frac{2}{3} p^2 t B_{p-3} \pmod{p^3}.$$

Proof. By the proof of Lemma 2.1, for $k \in \{1, 2, \dots, p-1\}$ we have $\binom{pt}{k} \binom{-1-pt}{k} \equiv -\frac{p^2 t^2}{k^2} - \frac{pt}{k} \pmod{p^3}$. Thus,

$$\sum_{k=1}^{p-1} \binom{pt}{k} \binom{-1-pt}{k} \frac{1}{k} \equiv -p^2 t^2 \sum_{k=1}^{p-1} \frac{1}{k^3} - pt \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^3}.$$

By [L] or [S2, Corollary 5.1], $\sum_{k=1}^{p-1} \frac{1}{k^3} \equiv 0 \pmod{p}$ and $\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2}{3} p B_{p-3} \pmod{p^2}$. Thus the result follows.

Lemma 5.2. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod{p}$ and $t = (a - \langle a \rangle_p)/p$. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{a}{k} \binom{-1-a}{k}}{k} \equiv -\frac{2}{3} p^2 t B_{p-3} - 2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} + 2pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} + 2p^2 t \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^3} \pmod{p^3}.$$

Proof. For $1 \leq k \leq \langle a \rangle_p$ we have $\langle a-k+1 \rangle_p = \langle a \rangle_p - k + 1$ and so $a-k+1 = \langle a-k+1 \rangle_p + pt$. Using (5.2) we see that

$$\begin{aligned} A_{p-1}(a) - A_{p-1}(a - \langle a \rangle_p) &= \sum_{k=1}^{\langle a \rangle_p} (A_{p-1}(a - k + 1) - A_{p-1}(a - k)) \\ &\equiv \sum_{k=1}^{\langle a \rangle_p} \left(\frac{2t(t+1)p^2}{\langle a-k+1 \rangle_p^3} - \frac{2}{a-k+1} \right) \\ &= 2t(t+1)p^2 \sum_{k=1}^{\langle a \rangle_p} \frac{1}{(\langle a \rangle_p - k + 1)^3} - 2 \sum_{k=1}^{\langle a \rangle_p} \frac{1}{\langle a \rangle_p - k + 1 + pt} \\ &= 2t(t+1)p^2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^3} - 2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r + pt} \pmod{p^3}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r + pt} &= \sum_{r=1}^{\langle a \rangle_p} \frac{r^2 - ptr + p^2 t^2}{r^3 - (pt)^3} \equiv \sum_{r=1}^{\langle a \rangle_p} \frac{r^2 - ptr + p^2 t^2}{r^3} \\ &= \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} - pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} + p^2 t^2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^3} \pmod{p^3}. \end{aligned}$$

We then obtain

$$A_{p-1}(a) - A_{p-1}(a - \langle a \rangle_p) \equiv -2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} + 2pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} + 2p^2 t \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^3} \pmod{p^3}.$$

By Lemma 5.1, $A_{p-1}(a - \langle a \rangle_p) = A_{p-1}(pt) \equiv -\frac{2}{3} p^2 t B_{p-3} \pmod{p^3}$. Thus the result follows.

Theorem 5.1. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod{p}$ and $t = (a - \langle a \rangle_p)/p$. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{a}{k} \binom{-1-a}{k}}{k} \equiv -\frac{2}{3} p^2 t(t+1) B_{p-3}(-a) - 2 \frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}.$$

Proof. It is well known that (see [MOS])

$$(5.3) \quad \sum_{r=1}^m r^k = \frac{B_{k+1}(m+1) - B_{k+1}}{k+1},$$

$$(5.4) \quad B_n(x+y) = \sum_{k=0}^n \binom{n}{k} y^k B_{n-k}(x), \quad B_n(1-x) = (-1)^n B_n(x).$$

Thus, using Euler's theorem we see that

$$\begin{aligned} & pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} - \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} \\ & \equiv pt \sum_{r=1}^{\langle a \rangle_p} r^{p^2(p-1)-2} - \sum_{r=1}^{\langle a \rangle_p} r^{p^2(p-1)-1} \\ & = pt \frac{B_{p^2(p-1)-1}(\langle a \rangle_p + 1) - B_{p^2(p-1)-1}}{p^2(p-1) - 1} - \frac{B_{p^2(p-1)}(\langle a \rangle_p + 1) - B_{p^2(p-1)}}{p^2(p-1)} \\ & = -pt \frac{B_{p^2(p-1)-1}(-\langle a \rangle_p)}{p^2(p-1) - 1} - \frac{B_{p^2(p-1)}(-\langle a \rangle_p) - B_{p^2(p-1)}}{p^2(p-1)} \\ & = -pt \frac{B_{p^2(p-1)-1}(pt - a)}{p^2(p-1) - 1} - \frac{B_{p^2(p-1)}(pt - a) - B_{p^2(p-1)}(-a) + B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \\ & = \frac{-pt}{p^2(p-1) - 1} \sum_{k=0}^{p^2(p-1)-1} \binom{p^2(p-1)-1}{k} (pt)^k B_{p^2(p-1)-1-k}(-a) \\ & \quad - \frac{1}{p^2(p-1)} \sum_{k=1}^{p^2(p-1)} \binom{p^2(p-1)}{k} (pt)^k B_{p^2(p-1)-k}(-a) - \frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}. \end{aligned}$$

By [S1, Lemma 2.3], $B_m(-a) \in \mathbb{Z}_p$ for $m \not\equiv 0 \pmod{p-1}$ and $pB_m(-a) \in \mathbb{Z}_p$ for $m \equiv$

0 (mod $p - 1$). Thus,

$$\begin{aligned}
& pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} - \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} + \frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \\
& \equiv pt(B_{p^2(p-1)-1}(-a) + (p^2(p-1) - 1)ptB_{p^2(p-1)-2}(-a)) \\
& \quad - ptB_{p^2(p-1)-1}(-a) - \frac{p^2(p-1) - 1}{2}(pt)^2B_{p^2(p-1)-2}(-a) \\
& \equiv -p^2t^2B_{p^2(p-1)-2}(-a) + \frac{1}{2}p^2t^2B_{p^2(p-1)-2}(-a) \\
& = -\frac{1}{2}p^2t^2B_{(p^2-1)(p-1)+p-3}(-a) \pmod{p^3}.
\end{aligned}$$

By [S2, Corollary 3.1],

$$B_{(p^2-1)(p-1)+p-3}(-a) \equiv ((p^2 - 1)(p - 1) + p - 3) \frac{B_{p-3}(-a)}{p - 3} \equiv \frac{2}{3}B_{p-3}(-a) \pmod{p}.$$

Thus,

$$(5.5) \quad pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} - \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} \equiv -\frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} - \frac{1}{3}p^2t^2B_{p-3}(-a) \pmod{p^3}.$$

By [S2, Lemma 3.2],

$$\sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^3} \equiv \sum_{r=1}^{\langle a \rangle_p} r^{p-4} \equiv \frac{B_{p-3}(-a) - B_{p-3}}{p - 3} \equiv -\frac{1}{3}(B_{p-3}(-a) - B_{p-3}) \pmod{p}.$$

Thus, from Lemma 5.2 and (5.5) we derive that

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{\binom{a}{k} \binom{-1-a}{k}}{k} & \equiv -\frac{2}{3}p^2tB_{p-3} - 2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} + 2pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} + 2tp^2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^3} \\
& \equiv -\frac{2}{3}p^2tB_{p-3} - 2 \frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \\
& \quad - \frac{2}{3}p^2t^2B_{p-3}(-a) - \frac{2}{3}p^2t(B_{p-3}(-a) - B_{p-3}) \\
& = -\frac{2}{3}p^2t(t+1)B_{p-3}(-a) - 2 \frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}.
\end{aligned}$$

This completes the proof.

Lemma 5.3 ([MOS]). *For any positive integer n we have*

$$B_{2n}\left(\frac{1}{2}\right) = (2^{1-2n} - 1)B_{2n}, \quad B_{2n}\left(\frac{1}{3}\right) = \frac{3 - 3^{2n}}{2 \cdot 3^{2n}}B_{2n},$$

$$B_{2n}\left(\frac{1}{4}\right) = \frac{2 - 2^{2n}}{4^{2n}}B_{2n}, \quad B_{2n}\left(\frac{1}{6}\right) = \frac{(2 - 2^{2n})(3 - 3^{2n})}{2 \cdot 6^{2n}}B_{2n}.$$

For an odd prime p and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$ let $q_p(a)$ be the Fermat quotient given by $q_p(a) = (a^{p-1} - 1)/p$. By Fermat's little theorem, $q_p(a) \in \mathbb{Z}_p$.

Theorem 5.2. *Let $p > 3$ be a prime. Then*

$$(i) \quad \sum_{k=1}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k k} \equiv 3q_p(3) - \frac{3}{2}pq_p(3)^2 + p^2q_p(3)^3 + \frac{52}{27}p^2B_{p-3} \pmod{p^3},$$

$$(ii) \quad \sum_{k=1}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{64^k k} \equiv 6q_p(2) - 3pq_p(2)^2 + 2p^2q_p(2)^3 + \frac{7}{2}p^2B_{p-3} \pmod{p^3},$$

$$(iii) \quad \sum_{k=1}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k k} \equiv 4q_p(2) + 3q_p(3) - p\left(2q_p(2)^2 + \frac{3}{2}q_p(3)^2\right)$$

$$+ p^2\left(\frac{4}{3}q_p(2)^3 + q_p(3)^3\right) + \frac{455}{54}p^2B_{p-3} \pmod{p^3}.$$

Proof. By Lemma 5.3,

$$B_{p-3}\left(\frac{1}{3}\right) = \frac{3 - 3^{p-3}}{2 \cdot 3^{p-3}}B_{p-3} \equiv 13B_{p-3} \pmod{p},$$

$$B_{p-3}\left(\frac{1}{4}\right) = \frac{2 - 2^{p-3}}{4^{p-3}}B_{p-3} \equiv 28B_{p-3} \pmod{p},$$

$$B_{p-3}\left(\frac{1}{6}\right) = \frac{(2 - 2^{p-3})(3 - 3^{p-3})}{2 \cdot 6^{p-3}}B_{p-3} \equiv 91B_{p-3} \pmod{p}.$$

By [S4, p.287],

$$(5.6) \quad \frac{B_{p^2(p-1)} - B_{p^2(p-1)}\left(\frac{1}{3}\right)}{p^2(p-1)} \equiv \frac{3}{2}\left(q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3\right) \pmod{p^3},$$

$$(5.7) \quad \frac{B_{p^2(p-1)} - B_{p^2(p-1)}\left(\frac{1}{4}\right)}{p^2(p-1)} \equiv 3\left(q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3\right) \pmod{p^3},$$

$$(5.8) \quad \frac{B_{p^2(p-1)} - B_{p^2(p-1)}\left(\frac{1}{6}\right)}{p^2(p-1)} \equiv 2\left(q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3\right)$$

$$+ \frac{3}{2}\left(q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3\right) \pmod{p^3}.$$

Now taking $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Theorem 5.1 and then applying (1.11) and the above we deduce the result.

Remark 5.1 Let $p > 3$ be a prime. In [T3] Tauraso gave a congruence for $\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \binom{-1-a}{k} \pmod{p^2}$, and showed that $\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k} \equiv -2H_{\frac{p-1}{2}} \pmod{p^3}$, which can be deduced from Theorem 5.1 (with $a = -\frac{1}{2}$) and the congruence ([S2, Theorem 5.2(c)])

$$H_{\frac{p-1}{2}} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Theorem 5.2(ii) is equivalent to Z.W. Sun's conjecture (1.10).

6. Congruences for $\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{a}{k} \pmod{p^2}$.

For given positive integer n and variables a and b define

$$f_n(a, b) = \sum_{k=1}^n \frac{\binom{a}{k}}{k \binom{b}{k}}.$$

Then

$$f_n(a, b) - f_n(a-1, b) = \sum_{k=1}^n \frac{\binom{a}{k} - \binom{a-1}{k}}{k \binom{b}{k}} = \sum_{k=1}^n \frac{\frac{1}{k} \binom{a-1}{k-1}}{\binom{b}{k}} = \frac{1}{a} \sum_{k=1}^n \frac{\binom{a}{k}}{\binom{b}{k}}.$$

By Lerch's theorem ([B, p.86]) or induction on n ,

$$(6.1) \quad \sum_{k=0}^n \frac{\binom{a}{k}}{\binom{b}{k}} = \frac{b+1}{b+1-a} \left\{ 1 - \frac{\binom{a}{n+1}}{\binom{b+1}{n+1}} \right\}.$$

Thus,

$$(6.2) \quad \begin{aligned} f_n(a, b) - f_n(a-1, b) &= \frac{1}{a} \left\{ \frac{b+1}{b+1-a} - 1 - \frac{b+1}{b+1-a} \cdot \frac{\binom{a}{n+1}}{\binom{b+1}{n+1}} \right\} \\ &= \frac{1}{b+1-a} - \frac{1}{b+1-a} \cdot \frac{\binom{a-1}{n}}{\binom{b}{n}}. \end{aligned}$$

Lemma 6.1. *Let $p > 3$ be a prime, $n \in \{1, 2, 3, \dots\}$, $a, b \in \mathbb{Z}_p$, $1 \leq \langle a \rangle_p \leq n \leq \langle b \rangle_p$ and $a = \langle a \rangle_p + pt$. Then*

$$\begin{aligned} \sum_{k=1}^n \frac{\binom{a}{k}}{k \binom{b}{k}} &\equiv pt \sum_{k=1}^n \frac{(-1)^{k-1}}{k^2 \binom{b}{k}} + \sum_{r=1}^{\langle a \rangle_p} \frac{1}{b+1-r} + pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{(b+1-r)^2} \\ &\quad - \frac{pt}{\binom{b}{n}} \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^{n-r}}{(b+1-r)r \binom{n}{r}} \pmod{p^2}. \end{aligned}$$

Proof. By (6.2),

$$\begin{aligned}
f_n(a, b) - f_n(a - \langle a \rangle_p, b) &= \sum_{k=1}^{\langle a \rangle_p} (f_n(a - k + 1, b) - f_n(a - k, b)) \\
&= \sum_{k=1}^{\langle a \rangle_p} \frac{1}{b + 1 - (a - k + 1)} - \sum_{k=1}^{\langle a \rangle_p} \frac{1}{b + 1 - (a - k + 1)} \cdot \frac{\binom{a-k+1-1}{n}}{\binom{b}{n}} \\
&= \sum_{k=1}^{\langle a \rangle_p} \frac{1}{b + 1 - a + \langle a \rangle_p - (\langle a \rangle_p - k + 1)} \\
&\quad - \frac{1}{\binom{b}{n}} \sum_{k=1}^{\langle a \rangle_p} \frac{1}{b + 1 - a + \langle a \rangle_p - (\langle a \rangle_p - k + 1)} \binom{\langle a \rangle_p - k + 1 + a - \langle a \rangle_p - 1}{n}.
\end{aligned}$$

Set $a = \langle a \rangle_p + pt$. Substituting k with $\langle a \rangle_p + 1 - r$ in the above we obtain

$$(6.3) \quad f_n(a, b) - f_n(pt, b) = \sum_{r=1}^{\langle a \rangle_p} \frac{1}{b + 1 - r - pt} - \frac{1}{\binom{b}{n}} \sum_{r=1}^{\langle a \rangle_p} \frac{1}{b + 1 - r - pt} \binom{r + pt - 1}{n}.$$

For $1 \leq r \leq \langle a \rangle_p$ we see that

$$\begin{aligned}
\binom{r - 1 + pt}{n} &= \frac{(r - 1 + pt)(r - 2 + pt) \cdots (r - n + pt)}{n!} \\
&= \frac{(r - 1 + pt)(r - 2 + pt) \cdots (1 + pt)pt(pt - 1) \cdots (pt - (n - r))}{n!} \\
&\equiv \frac{(r - 1)! \cdot pt \cdot (-1)^{n-r} \cdot (n - r)!}{n!} \\
&= (-1)^{n-r} \frac{pt}{r \binom{n}{r}} \pmod{p^2}
\end{aligned}$$

and so

$$\begin{aligned}
f_n(a, b) - f_n(pt, b) &\equiv \sum_{r=1}^{\langle a \rangle_p} \frac{1}{b + 1 - r - pt} - \frac{1}{\binom{b}{n}} \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^{n-r}}{b + 1 - r - pt} \cdot \frac{pt}{r \binom{n}{r}} \\
&\equiv \sum_{r=1}^{\langle a \rangle_p} \frac{b + 1 - r + pt}{(b + 1 - r)^2} - \frac{pt}{\binom{b}{n}} \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^{n-r}}{(b + 1 - r)r \binom{n}{r}} \pmod{p^2}.
\end{aligned}$$

On the other hand,

$$\sum_{k=1}^n \frac{\binom{pt}{k}}{k \binom{b}{k}} = \sum_{k=1}^n \frac{pt \binom{pt-1}{k-1}}{k^2 \binom{b}{k}} \equiv pt \sum_{k=1}^n \frac{\binom{-1}{k-1}}{k^2 \binom{b}{k}} = pt \sum_{k=1}^n \frac{(-1)^{k-1}}{k^2 \binom{b}{k}} \pmod{p^2}.$$

Thus, the result follows.

Theorem 6.1. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$ and $a \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{a}{k} \equiv \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(-a)}{p^2(p-1)} + \frac{a - \langle a \rangle_p}{2} B_{p-2}(-a) \pmod{p^2}.$$

Proof. Set $a = \langle a \rangle_p + pt$. As $\binom{-1}{k} = (-1)^k$, taking $b = -1$ and $n = p - 1$ in Lemma 6.1 and then applying the well known fact $\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}$, (5.5) and (3.6) we see that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{a}{k} &= \sum_{k=1}^{p-1} \frac{\binom{a}{k}}{k \binom{-1}{k}} \equiv -pt \sum_{k=1}^{p-1} \frac{1}{k^2} - \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r} + pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} + pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2} \\ &\equiv \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(-a)}{p^2(p-1)} + \frac{pt}{2} B_{p-2}(-a) \pmod{p^2}. \end{aligned}$$

This proves the theorem.

Remark 6.1 In [T2] Tauraso showed that for any prime $p > 3$,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/2}{k} \equiv -H_{\frac{p-1}{2}} \pmod{p^3}.$$

In [L] Lehmer proved that $H_{\frac{p-1}{2}} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}$. Thus,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/2}{k} \equiv 2q_p(2) - pq_p(2)^2 \pmod{p^2}.$$

This can be deduced from Theorem 6.1 (with $a = -\frac{1}{2}$) and Lemma 5.3.

Theorem 6.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/4}{k} \equiv 3q_p(2) - \frac{3}{2}pq_p(2)^2 - \left(2 - \left(\frac{-1}{p}\right)\right)pE_{p-3} \pmod{p^2}.$$

Proof. Taking $a = -\frac{1}{4}$ in Theorem 6.1 we see that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/4}{k} \equiv \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} + \frac{-\frac{1}{4} - \langle -\frac{1}{4} \rangle_p}{2} B_{p-2}\left(\frac{1}{4}\right) \pmod{p^2}.$$

By (5.7), $\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} \equiv 3q_p(2) - \frac{3}{2}pq_p(2)^2 \pmod{p^2}$. By the proof of Theorem 3.2, $B_{p-2}(\frac{1}{4}) \equiv 8E_{p-3} \pmod{p}$. Now, from the above and (2.3) we deduce the result.

Theorem 6.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/3}{k} \equiv \frac{3}{2}q_p(3) - \frac{3}{4}pq_p(3)^2 - \frac{3 - (\frac{-3}{p})}{2}pU_{p-3} \pmod{p^2}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 6.1 we see that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/3}{k} \equiv \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{3})}{p^2(p-1)} + \frac{-\frac{1}{3} - \langle -\frac{1}{3} \rangle_p}{2} B_{p-2}(\frac{1}{3}) \pmod{p^2}.$$

By (5.6), $\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{3})}{p^2(p-1)} \equiv \frac{3}{2}q_p(3) - \frac{3}{4}pq_p(3)^2 \pmod{p^2}$. By [S7, p.217], $B_{p-2}(\frac{1}{3}) \equiv 6U_{p-3} \pmod{p}$. Now, from the above and (2.3) we deduce the result.

Theorem 6.4. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-1/6}{k} \\ & \equiv 2q_p(2) + \frac{3}{2}q_p(3) - pq_p(2)^2 - \frac{3}{4}pq_p(3)^2 - \frac{5}{2} \left(3 - 2 \left(\frac{-3}{p} \right) \right) pU_{p-3} \pmod{p^2}. \end{aligned}$$

Proof. Taking $a = -\frac{1}{6}$ in Theorem 6.1 and then applying (5.8), (2.3) and the fact $B_{p-2}(\frac{1}{6}) \equiv 30U_{p-3} \pmod{p}$ we deduce the result.

7. Congruences for $\sum_{k=0}^{p-1} \binom{a}{k} (-2)^k \pmod{p^2}$.

For given positive integer n and variables a and x define

$$F_n(a, x) = \sum_{k=0}^n \binom{a}{k} x^k.$$

Then

$$\begin{aligned} F_n(a, x) - (x+1)F_n(a-1, x) &= \sum_{k=0}^n \left(\binom{a}{k} - \binom{a-1}{k} \right) x^k - \sum_{k=0}^n \binom{a-1}{k} x^{k+1} \\ &= \sum_{k=1}^n \binom{a-1}{k-1} x^k - \sum_{k=0}^n \binom{a-1}{k} x^{k+1}. \end{aligned}$$

Thus,

$$(7.1) \quad F_n(a, x) - (x+1)F_n(a-1, x) = -\binom{a-1}{n} x^{n+1}.$$

Suppose that $p > 3$ is a prime, $a \in \mathbb{Z}_p$ and $a = \langle a \rangle_p + pt$. Taking $n = p - 1$ in (7.1) and then applying Lemma 3.1 we see that

$$\begin{aligned} & F_{p-1}(a, x) - (x+1)F_{p-1}(a-1, x) \\ &= -\binom{\langle a \rangle_p + pt - 1}{p-1} x^p \equiv \left(-\frac{pt}{\langle a \rangle_p} + \frac{p^2 t^2}{\langle a \rangle_p^2} - \frac{p^2 t}{\langle a \rangle_p} H_{\langle a \rangle_p} \right) x^p \pmod{p^3}. \end{aligned}$$

For $1 \leq k \leq \langle a \rangle_p$ we have $\langle a - k + 1 \rangle_p = \langle a \rangle_p - k + 1$ and so $a - k + 1 = \langle a - k + 1 \rangle_p + pt$. Thus,

$$\begin{aligned} & F_{p-1}(a, x) - (x+1)^{\langle a \rangle_p} F_{p-1}(a - \langle a \rangle_p, x) \\ &= \sum_{k=1}^{\langle a \rangle_p} (x+1)^{k-1} (F_{p-1}(a - k + 1, x) - (x+1)F_{p-1}(a - k, x)) \\ &\equiv \sum_{k=1}^{\langle a \rangle_p} (x+1)^{k-1} x^p \left(-\frac{pt}{\langle a \rangle_p - k + 1} + \frac{p^2 t^2}{(\langle a \rangle_p - k + 1)^2} - \frac{p^2 t}{\langle a \rangle_p - k + 1} H_{\langle a \rangle_p - k + 1} \right) \\ &= x^p \sum_{r=1}^{\langle a \rangle_p} (x+1)^{\langle a \rangle_p - r} \left(-\frac{pt}{r} + \frac{p^2 t^2}{r^2} - \frac{p^2 t}{r} H_r \right) \pmod{p^3} \end{aligned}$$

and therefore

$$\begin{aligned} & (7.2) \\ & F_{p-1}(a, x) - (x+1)^{\langle a \rangle_p} F_{p-1}(pt, x) \\ &\equiv x^p (x+1)^{\langle a \rangle_p} \left(-pt \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r(x+1)^r} + p^2 t^2 \sum_{r=1}^{\langle a \rangle_p} \frac{1}{r^2(x+1)^r} - p^2 t \sum_{r=1}^{\langle a \rangle_p} \frac{H_r}{r(x+1)^r} \right) \pmod{p^3}. \end{aligned}$$

Theorem 7.1. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$ and $a \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} (-2)^k \equiv (-1)^{\langle a \rangle_p} - (a - \langle a \rangle_p) E_{p-2}(-a) \pmod{p^2}.$$

Proof. Set $t = (a - \langle a \rangle_p)/p$. Taking $x = -2$ in (7.2) we see that

$$\sum_{k=0}^{p-1} \binom{a}{k} (-2)^k - (-1)^{\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{pt}{k} (-2)^k \equiv 2^p (-1)^{\langle a \rangle_p} pt \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r} \pmod{p^2}.$$

By a result of Glaisher (see [S4]), $\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2q_p(2) \pmod{p}$. Note that $\binom{pt-1}{k-1} \equiv \binom{-1}{k-1} = (-1)^{k-1} \pmod{p}$ for $1 \leq k \leq p-1$. We then derive that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{pt}{k} (-2)^k &= 1 + pt \sum_{k=1}^{p-1} \binom{pt-1}{k-1} \frac{(-2)^k}{k} \\ &\equiv 1 - pt \sum_{k=1}^{p-1} \frac{2^k}{k} \equiv 1 + 2ptq_p(2) \pmod{p^2}. \end{aligned}$$

It is well known that $pB_{p-1} \equiv p-1 \pmod{p}$. Thus, from Lemma 2.2 we deduce that

$$\begin{aligned} \sum_{r=1}^{\langle a \rangle_p} \frac{(-1)^r}{r} &\equiv -\frac{q_p(2)pB_{p-1}}{p-1} + \frac{1}{2}(-1)^{\langle a \rangle_p+1}E_{p-2}(-a) \\ &\equiv -q_p(2) - \frac{1}{2}(-1)^{\langle a \rangle_p}E_{p-2}(-a) \pmod{p}. \end{aligned}$$

Now combining all the above we deduce that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{a}{k} (-2)^k &\equiv (-1)^{\langle a \rangle_p} (1 + 2ptq_p(2)) + 2(-1)^{\langle a \rangle_p} pt \left(-q_p(2) - \frac{1}{2}(-1)^{\langle a \rangle_p}E_{p-2}(-a) \right) \\ &= (-1)^{\langle a \rangle_p} - ptE_{p-2}(-a) \pmod{p^2}. \end{aligned}$$

This proves the theorem.

Theorem 7.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-1/3}{k} (-2)^k \equiv \left(\frac{-3}{p} \right) + \frac{3 - (\frac{-3}{p})}{3} pq_p(2) \pmod{p^2}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 7.1 and applying (2.3) we see that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-1/3}{k} (-2)^k &\equiv (-1)^{\langle -\frac{1}{3} \rangle_p} - \left(-\frac{1}{3} - \langle -\frac{1}{3} \rangle_p \right) E_{p-2}\left(\frac{1}{3}\right) \\ &= \left(\frac{-3}{p} \right) - \frac{(\frac{-3}{p}) - 3}{6} pE_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}. \end{aligned}$$

By [S7, Lemma 2.2], Lemma 5.3 and the fact $pB_{p-1} \equiv p-1 \pmod{p}$,

$$\begin{aligned} E_{p-2}\left(\frac{1}{3}\right) &= \frac{2}{p-1}((-2)^{p-1} - 1)B_{p-1}\left(\frac{1}{3}\right) = \frac{2}{p-1} \cdot pq_p(2) \cdot \frac{3 - 3^{p-1}}{2 \cdot 3^{p-1}} B_{p-1} \\ &\equiv 2q_p(2) \pmod{p}. \end{aligned}$$

Thus the result follows.

Remark 7.1 In [Su1], Z.W. Sun proved that for any prime $p > 3$,

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{2^k} = \sum_{k=1}^{p-1} \binom{-1/2}{k} (-2)^k \equiv \left(\frac{-1}{p} \right) - p^2 E_{p-3} \pmod{p^3}.$$

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